150A Homework Solutions

Homework 1

- Section 1.2: 2, 3
- Section 1.3: *8
- Section 1.4: 1b, 2, 10, *11
- Section 1.5: *1, *2, 7, 8, 11

Section 1.2

- Problem 2 α closest to origin means that $\|\alpha\| = \sqrt{\alpha \cdot \alpha}$ is minimal, that is $\alpha \cdot \alpha$ is minimal, therefore $0 = (\alpha \cdot \alpha)' = 2\alpha \cdot \alpha'$ so $\alpha \perp \alpha'$.
- Problem 3 If $\alpha'' = 0$ then by integration $\alpha'(t) = ut + v$ with $u, v \in \mathbb{R}^3$ are constant vectors, so α is a straight line (see do Carmo page 16)

Section 1.3

- Problem 8 See Rudin's "Principles of Mathematical Analysis" pages 136, 137.

Section 1.4

- Problem 1b The basis shown can be obtained by multiplying the standard basis by the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{pmatrix}$ which has determinant +39, so it is positive.

  By row reduction:

  $$\det \begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -4 \\ 0 & -3 & -17 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -4 \\ 0 & 0 & -13 \end{pmatrix} = 39$$

- Problem 2 $P = \{(x, y, z) \colon ax + by + cz + d = 0\}$

  Assume first that $d = 0$. The position vector (x, y, z) for any point on this plane satisfies $au + bv + cw = 0$. Therefore $(a, b, c) \cdot (x, y, z) = au + bv + cw = 0$ so (a, b, c) is perpendicular to (x, y, z) and therefore to the plane P. Since adding a constant displaces the plan parallel to itself, this still holds for arbitrary d.

  Since $v = (a, b, c)$ is perpendicular to P, the distance from the origin to P must be $|rv| = |r||v|$ for some $r \in \mathbb{R}$.

  To find r, we must have rv lie in P. That is, $a(ra) + b(rb) + c(rc) + d = 0$ or $v \cdot (rv) = rv \cdot v = -d$ so $r = \frac{-d}{v \cdot v}$. Notice this undefined when $v \cdot v = 0$ which happens when $d = 0$ but then the distance is clearly zero.

  Therefore distance $= |r||v| = |r|\sqrt{v \cdot v} = \frac{|d|}{v \cdot v}\sqrt{v \cdot v} = \frac{|d|}{\sqrt{v \cdot v}} = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

  Key Point:
  • The "distance vector" must be perpendicular to the plane

- Problem 10 Page 13 shows that $\det(S) = A^2$ where $S = \begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix}$ and A is the area of the parallelogram with edges $u, v \in \mathbb{R}^2$.

  Notice that $S = RR^T$ where $R = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$ by the definition of matrix multiplication and dot product. For example, $u \cdot v = (u_1 e_1 + u_2 e_2) \cdot (v_1 e_1 + v_2 e_2) = u_1 v_1 + u_2 v_2$ since $e_1 \cdot e_1 = e_2 \cdot e_2 = 1$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$

  Therefore, $A^2 = \det S = \det(RR^T) = (\det R)(\det R^T) = (\det R)(\det R) = \det R^2$ since $\det R = \det R^T$.

- Problem 11

  (a) The volume V of a parallelepiped is $V = A \cdot h$ where A is the area of one face and h is the height measured perpendicularly to that face. If u and v define the face, and w is at angle θ with the perpendicular $u \times v$, then $|w| \cos\theta$ gives the height. Since $|u \times v|$ gives the area of the face, we have $\pm V = |u \times v||w| \cos\theta = (u \times v) \cdot w$ (page 4: $u \cdot v = |u||v| \cos\theta$).

  (b) From page 12 we get $\det(u, v, w) = (u \times v) \cdot w$ and arguing as in problem 10 we have $V^2 = \det(R)^2 = \det(RR^T) = \det(S)$ where R is the matrix (u, v, w) and S is as in the exercise.

Section 1.5

- Problem 1 $\alpha(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c})$ with $c^2 = a^2 + b^2$

  Note that we need to assume that $a \neq 0$ for most of this problem.

  $\alpha'(s) = (-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c})$ $\qquad \alpha''(s) = (-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0)$

  (a) $\|\alpha'\|^2 = \frac{a^2}{c^2}(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c}) + \frac{b^2}{c^2} = \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$

  (b) $k = |\alpha''| = \sqrt{\frac{a^2}{c^4}} = a/c^2$

  $b = t \times n = (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c})$

  $b' = (\frac{b}{c^2} \cos \frac{s}{c}, \frac{b}{c} \sin \frac{s}{c}, 0)$

  and since $b' = -\frac{b}{c^2} \cdot n = \tau n$ we have $\tau = -\frac{b}{c^2}$

  (c) The osculating plane is the plane determined by t and n (or α' and α''; and they must be nonzero) at $\alpha(s)$. It is the set of all points p so that $p - \alpha(s)$ is orthogonal to $t \times n$, that is

  $P = \{p \in \mathbb{R}^3 \colon t \times n \perp p - \alpha(s)\} = \{p \in \mathbb{R}^3 \colon t \times n \cdot (p - \alpha(s)) = 0\}$.

  which is the set of all points p so that the $\det(t, n, p - \alpha(s)) = 0$

  Since $n = \alpha''/k = (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0)$ we get

  $xbc \sin \frac{s}{c} - ybc \sin \frac{s}{c} + zac - abs = 0$

  or

  $xb \sin \frac{s}{c} - yb \sin \frac{s}{c} + za - ab\frac{s}{c} = 0$

  (d) We need to show that $(0, 0, 1) \perp n(s)$, that is $(0, 0, 1) \cdot (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0) = 0$ which is obvious.

(e) We need to show that $(0,0,1)\cdot t(s)$ is constant. This is clear both from the expression for t which gives $(0,0,1)\cdot t = \frac{b}{c}$ and from the problem above since $(0,0,1)\cdot t$ constant means $0 = ((0,0,1)\cdot t)' = (0,0,1)\cdot t' = k((0,0,1)\cdot n)$ since we know $k \neq 0$ (assuming $a \neq 0$)

- Problem 2

$\tau = b' \cdot n$	since $b' \cdot n = \tau n \cdot n$
$= (t \times n)' \cdot n$	since $b = t \times n$
$= (t \times n') \cdot n$	since $t' \times n = kn \times n = 0$
$= -t \times n \cdot n'$	since $u \times v \cdot w = -u \times w \cdot v$
$= -\alpha' \times \frac{\alpha''}{k} \cdot (\frac{\alpha''}{k})'$	since $t = \alpha'$ and $n = \alpha''/k$
$= -\alpha' \times \frac{\alpha''}{k} \cdot (\frac{\alpha'''}{k} + \alpha''(\frac{1}{k})')$	
$= -\alpha' \times \frac{\alpha''}{k} \cdot \frac{\alpha'''}{k}$	since $= \alpha' \times \frac{\alpha''}{k} \cdot \alpha''(\frac{1}{k})' = 0$
$= -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{k^2}$	

Homework 2

- Okikiolu's problem
- Section 2.2: 1, 2, *4, *5, *15

Okikiolu's problem

Show that if a map T from 3-space to itself preserves distance, that is $\forall u(|Tu - Tv| = |u - v|)$, then T has the form $Tu = Ou + w$, where w is a 3-vector and O is a linear orthogonal transformation of 3-space.

O is an orthogonal transformation iff $\forall u, v(Ou \cdot Ov = u \cdot v)$.

$$
\begin{aligned}
&(Tu - T0) \cdot (Tv - T0) \\
&= Tu \cdot Tv - Tu \cdot T0 - Tv \cdot T0 + T0 \cdot T0 \\
&= \tfrac{|Tu|^2 + |Tv|^2 - |Tu - Tv|}{2} \\
&\quad + \tfrac{-|Tu|^2 - |T0|^2 + |Tu - T0|}{2} \\
&\quad + \tfrac{-|Tv|^2 - |T0|^2 + |Tv - T0|}{2} + |T0|^2 && \text{Since } a \cdot b = |a|^2 + |b|^2 - |a - b|^2 \\
&= -|u - v|^2 + |u|^2 + |v|^2 && \text{Since, e.g., } |Tu - T0| = |u - 0| = |u| \\
&= u \cdot v && \text{Since } a \cdot b = |a|^2 + |b|^2 - |a - b|^2
\end{aligned}
$$

Therefore, we can take $w = T0$ and $Ou = Tu - w$ is orthogonal.

Section 2.2

- Problem 1 Take $\phi(\theta, z) = (\sin\theta, \cos\theta, z)$. Then $\phi'(\theta, z) = \begin{pmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \\ 0 & 1 \end{pmatrix}$.

  Its two columns are clearly linearly independent for every θ and z, so ϕ is a coordinate map for every point on the cylinder. ϕ is a smooth with smooth inverses $\phi^{-1}(x, y, z) = (\arctan(x/y), z) + k\pi \quad (k \in \mathbb{Z})$ (these are compositions of known smooth functions).

- Problem 2 The first set is not a regular surface since the points with $x^2 + y^2 = 1$ (i.e., on the boundary) fail point 2 of the definition. The second set is a regular surface: the coordinate map $\phi(x, y) = (x, y, 0)$ is clearly smooth with smooth inverse and $\phi'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

- Problem 3 The two-sheeted cone has a singularity at (0,0,0). This is not so easy to prove. Consider any neighborhood U induced on it by a neighborhood of $\mathbb{R}^3$. If we remove the point (0,0,0) from this neighborhood, we get two disconnected sets. However, any neighborhood V of $\mathbb{R}^2$ with one point removed is connected, hence no such V can not be homeomorphic to U and we fail condition 2 of the definition.

- Problem 4 $f(x, y, z) = z^2$ so $f'(x, y, z) = (0, 0, 2z)$. Notice that $(0, 0, 0) \in f^{-1}[\{0\}] = \mathbb{R}^2$ and (0,0,0) is a singular point since $f'(0, 0, 0) = (0, 0, 0)$. Hence 0 is not a regular value of f. But clearly $\mathbb{R}^2$ is a regular surface.

- Problem 5 The function given (I'll call it ϕ since I don't like $\mathbf{x}$) $\phi(u,v) = (u+v, u+v, uv)$ is a parametrization of P, since it is smooth (composition of smooth functions) with smooth inverse (by prop. 2.2.4 since $P \approx \mathbb{R}^2$ is regular) $\phi^{-1}(x,y,z) = (\frac{x+\sqrt{x^2-4z}}{2}, \frac{x-\sqrt{x^2-4z}}{2})$ (we get this from solving the equations: $x = u+v$ and $z = uv$ and by choosing the signs to satisfy $u > v$). We have $\phi'(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ v & u \end{pmatrix}$ which has rank two since $u > v$, so ϕ is regular. Finally since $\{(u,v) \in \mathbb{R}^2 \colon u > v\}$ is homeomorphic to $\mathbb{R}^2$, ϕ is a parameterization of P.
- Problem 15 For a given t, the equation of the line going from $(0,0,t)$ to $(a,t,0)$ can be given by $(sa, st, (1-s)t)$ so the set of points formed by these lines is $\phi[\mathbb{R}^2]$ where $\phi \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is given by $\phi(s,t) = (sa, st, (1-s)t)$. ϕ is clearly a local homeomorphism and $\phi'(s,t) = \begin{pmatrix} a & 0 \\ t & s \\ -t & 1-s \end{pmatrix}$ is regular since $a \neq 0$.

  Therefore, the surface so defined is regular.

Homework 3

- Section 2.3: 1, 2, 3, 4, 5
- Problem 1 Clearly A is a smooth bijection, it maps the sphere onto the sphere, and it is its own inverse.
- Problem 2 π is smooth by the comments in example 1 on page 72, since $\pi \subseteq \pi'$ where π' is the projection from $\mathbb{R}^3$ onto $\mathbb{R}^2$.
- Problem 3 Take $\phi(x, y) = (x, y, x^2 + y^2)$. Clearly, ϕ is smooth (it is a parametrization of the paraboloid, by definition) and its inverse is the projection which we saw above is smooth too.
- Problem 4 $\phi(x, y, z) = (ax, by, cz)$ maps the sphere into the ellipsoid smoothly and injectively with inverse $\phi(x, y, z) = (x/a, y/b, z/c)$. (We can assume $a, b, c \neq 0$ from the statement of the problem.)
- Problem 5 $d(p) = |p - p_0|$ is clearly smooth from $\mathbb{R}^3$ to $\mathbb{R}$ except at $p = p_0$. Since $p_0 \notin S$, $d(p)$ is smooth from S to $\mathbb{R}$ by the comments in example 1 on page 72.

Homework 4

- Section 2.4: 1, 2, 3, 12

  When a surface is given by $f^{\dashv}[\{a\}]$ for a regular value a of some smooth function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$, we have that for any curve α on that surface, $f(\alpha(t)) = a$ and therefore $df \circ d\alpha = 0$. If we choose an orthonormal basis, then df and $d\alpha$ are given, respectively, by a row and a column vector and $Df \cdot D\alpha = 0$ where the multiplication is just matrix multiplication. This is the same as saying $(Df)^T \cdot D\alpha = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) \cdot d\alpha/dt = 0$ (dot product). Therefore, any tangent vector is orthogonal to $(Df)^T$ so $(Df)^T$ is a vector normal to the tangent plane. We will use this for several of the problems (where Df stands for the matrix of the differential in the standard basis of $\mathbb{R}^n$).

- Problem 1 From above, the equation of the plane at $p_0 = (x_0, y_0, z_0)$ is just

  $(Df)^T \cdot (p - p_0) = 0$ since $(Df)^T$ is normal to the tangent plane at p_0

  Writing it out in more detail we have:

  $\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$

- Problem 2 Take $f = x^2 + y^2 - z^2$. The normal to the plane tangent to $f^{\dashv}[\{1\}]$ (our surface) at (x, y, z) is $(Df)^T = (2x, 2y, -2z)$. At any point with $z = 0$ this is $(2x, 2y, 0)$ which is orthogonal to $(0, 0, 1)$ (the z-axis). Since the normal of the tangent plane is orthogonal to the z-axis, the tangent plane is parallel to it.

- Problem 3 We can do this two ways.

  (a) Define $g(x, y, z) = z - f(x, y)$. By problem 1, the tangent plane at (x_0, y_0, z_0) is:

  $-\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + 1(z - z_0) = 0$

  Since $z_0 = f(x_0, y_0)$, we get:

  $\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + f(x_0, y_0) = z$

  (b) Use $\phi(x, y) = (x, y, f(x, y))$, which is a parametrization since it is smooth and $D\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial f/\partial x & \partial f/\partial y \end{pmatrix}$ always has rank 2. Two basis vectors for the tangent space are $D\phi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \partial f/\partial x \end{pmatrix} = u$ and $D\phi \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \partial f/\partial y \end{pmatrix} = v$, so a normal vector to the tangent space (and therefore also to the tangent plane) is $n = u \times v = \begin{pmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ 1 \end{pmatrix}$.

  Now that we have a normal vector, we proceed as in part (a).

- Problem 12 Define the functions $f(x,y,z) = x^2 + y^2 + z^2 - ax$, $g(x,y,z) = x^2 + y^2 + z^2 - by$, and $h(x,y,z) = x^2 + y^2 + z^2 - cz$.

  Then our three surfaces are $R = f^{\dashv}[\{0\}]$, $S = f^{\dashv}[\{0\}]$, and $T = f^{\dashv}[\{0\}]$.

  The corresponding derivatives are: $Df = (2x - a, 2y, 2z)$, $Dg = (2x, 2y - b, 2z)$, and $Dh = (2x, 2y, 2z - c)$. These can only be zero at $(a/2, 0, 0)$, $(0, b/2, 0)$, and $(0, 0, c/2)$ respectively, but these three points are not on the respective surfaces (i.e. $f(a/2, 0, 0) = a^2/4 - a^2/2 \neq 0$ since we are given $a \neq 0$ and similarly for the other two). The functions are clearly smooth and so by 2.2.2, R, S, T are regular surfaces. The normals to their tangent planes are given by the transposes of these derivatives as we saw above.

  On $R \cap S = \{(x,y,z)\colon f(x,y,z) = g(x,y,z) = 0\}$ we have

  $(Df)^T \cdot (Dg)^T = (2x - a)2x + 2y(2y - b) + 4z^2 = 4x^2 + 4y^2 + 4z^2 - 2ax - 2by = 2f(x,y,z) + 2g(x,y,z) = 0$ since on $R \cap S$ we have $f(x,y,z) = g(x,y,z) = 0$.

  Since their normals are orthogonal, the tangent planes of R and S intersect orthogonally, and that means that the surfaces R and S intersect orthogonally.

  The same argument works for $R \cap T$ and $S \cap T$.

Homework 5

- Section 2.5: 1a, 1c, 2, 4, 7, 11
- Problem 1a First fund. form for $x(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$

  $$Dx = \begin{pmatrix} a \cos u \cos v & -a \sin u \sin v \\ b \cos u \sin v & b \sin u \cos v \\ -c \sin u & 0 \end{pmatrix}$$

  Define $r = Dx \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $s = Dx \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

  Then the first fundamental form is given by the matrix

  $\begin{pmatrix} \langle r, r \rangle & \langle r, s \rangle \\ \langle s, r \rangle & \langle s, s \rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where

  $E = a^2 \cos^2 u \cos^2 v + b^2 \cos^2 u \sin^2 v + c^2 \sin^2 u$

  $F = (b^2 - a^2) \sin u \cos u \sin v \cos v$

  $G = a^2 \sin^2 u \sin^2 v + b^2 \sin^2 u \cos^2 v$

- Problem 1c First fund. form for $x(u,v) = (au \cosh v, bu \sinh v, u^2)$

  $$Dx = \begin{pmatrix} a \cosh v & au \sinh v \\ b \sinh v & bu \cosh v \\ 2u & 0 \end{pmatrix}$$

  $E = a^2 \cosh^2 v + b^2 \sinh^2 v + 4u^2$

  $F = (a^2 + b^2) u \cosh v \sinh v$

  $G = a^2 u^2 \sinh^2 v + b^2 u^2 \cosh^2 v$

- Problem 2 Find $\cos\beta$ where β is the acute angle of a curve with range $P \cap S^2$ and the semimeridian $\phi = \phi_0$ where S^2 is parametrized by $r(\phi,\theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $P = \{(x,y,z) \in \mathbb{R}^3 \colon x = kz\}$ where $k = \operatorname{cotan}\alpha$.

  Using the parametrization we have:

  $P \cap S^2 = P \cap \mathsf{R}r = \{(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \colon \sin\theta\cos\phi = k\cos\theta\} = \{(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \colon \cos\phi = k\operatorname{cotan}\theta\}$

  So we want a curve with $\phi = \arccos(k\operatorname{cotan}\theta)$, for example,

  $\alpha(\theta) = (\arccos(k\operatorname{cotan}\theta), \theta)$

  with tangent vector $\alpha'(\theta) = (\frac{k}{\sin^2\theta\sqrt{1-k^2\operatorname{cotan}^2\theta}}, 1) = (\frac{1}{\sin\theta\sqrt{k^2\sin^2\theta-\cos^2\theta}}, 1)$

  A semimeridian curve is given by $\beta(\theta) = (\phi_0, \theta)$ with tangent $\beta'(\theta) = (0,1)$.

  Where they intersect we have $\beta = \frac{\langle\alpha',\beta'\rangle}{|\alpha'||\beta'|}$

  where the inner product is given by the first fundamental form.

  $$Dr = \begin{pmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \sin\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\theta \end{pmatrix}$$

  so

  $E = \sin^2\theta\sin^2\phi + \sin^2\theta + \cos^2\phi = \sin^2\theta$

  $F = -\sin\theta\cos\theta\sin\phi\cos\phi + \sin\theta\cos\theta\sin\phi\cos\phi = 0$

  $G = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = 1$

  Since $F = 0$ the calculations are easy. We get $\langle\beta',\beta'\rangle = 1$ and $\langle\alpha',\beta'\rangle = 1$.

  $\langle\alpha',\alpha'\rangle = \sin^2\theta\frac{1}{(\sin\theta\sqrt{k^2\sin^2\theta-\cos^2\theta})^2} + 1 = \frac{1}{k^2\sin^2\theta-\cos^2\theta}$

  and so $\beta = \frac{\langle\alpha',\beta'\rangle}{|\alpha'||\beta'|} = \sqrt{k^2\sin^2\theta - \cos^2\theta}$

  Since the intersection will happen when $\arccos(k\operatorname{cotan}\theta) = \phi_0$ we need to have $\operatorname{cotan}\theta = \frac{1}{k}\cos\phi_0$ so $\theta = \arctan\frac{k}{\cos\phi_0}$

  Plugging this into what we got above, we get $\cos\beta = \sqrt{\frac{1+k^2/\cos^2\phi_0}{1+k^4/\cos^2\phi_0}}$

  (The "acute angle" part tells us to select the positive square root.)

- Problem 4 Given a surface parametrized by

  $x(u, v) = (u \cos v, u \sin v, \log \cos v + u)$ (for $-\pi/2 < v < \pi/2$)

  show that the two curves $x(u_1, \bullet)$ and $x(u_2, \bullet)$ determine arcs of equal length on all curves $x(\bullet, v_0)$

  $$Dx = \begin{pmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \\ 1 & -\tan v \end{pmatrix}$$

  so

  $E = \cos^2 v + \sin^2 v + 1 = 2$

  $F = -u \cos v \sin v + u \cos v \sin v - \tan v = -\tan v$

  $G = u^2 \sin^2 v + u^2 \cos^2 v + \tan^2 v = u^2 + \tan^2 v$

  The length of the curve α where $\alpha(u) = x(u, v_0)$ between u_1 and u_2 is given by:

  $\int_{u_1}^{u_2} \sqrt{E \frac{\partial u}{\partial u}} du = \int_{u_1}^{u_2} \sqrt{2} du = \sqrt{2}(u_2 - u_1)$ since $\frac{\partial v_0}{\partial u} = 0$.

  Clearly this value does not depend on v_0, which is what we need to show.

- Problem 7 Show that the curves of a parametrization x are a Tchebyshef net iff $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$

  The length of any two curves $x(\bullet, v_1)$ and $x(\bullet, v_2)$ between u_1 and u_2 are given by

  $\int_{u_1}^{u_2} \sqrt{E(u, v_1) \frac{\partial u}{\partial u}} du = \int_{u_1}^{u_2} \sqrt{E(u, v_1)} du$

  $\int_{u_1}^{u_2} \sqrt{E(u, v_2) \frac{\partial u}{\partial u}} du = \int_{u_1}^{u_2} \sqrt{E(u, v_2)} du$

  But since $\frac{\partial E}{\partial v} = 0$ we must have $E(u, v_1) = E(u, v_2)$ so these two lengths are equal. A similar argument shows that the length of any two curves $x(u_1, \bullet)$ and $x(u_2, \bullet)$ between v_1 and v_2 is equal when $\frac{\partial G}{\partial u} = 0$.

  Now assume $\frac{\partial E}{\partial v}(u_0, v_1) \neq 0$ for some u_0, v_1. Since E is smooth, there must be some v_2 so $E(u, v_1) \neq E(u, v_2)$. Assume $E(u, v_1) < E(u, v_2)$. Then since E is smooth, we must have u_1, u_2 with $u_1 < u_0 < u_2$ so that $E(u, v_1) < E(u, v_2)$ for all $u \in [u_1, u_2]$. Therefore, the length of the curve $x(\bullet, v_1)$ must be less than the length of the curve $x(\bullet, v_2)$ between u_1 and u_2. If instead, $E(u, v_2) < E(u, v_1)$ we get the opposite inequality. Either way, the curves of the parametrization are not a Tchebysef net. A similar argument shows that if $\frac{\partial G}{\partial u}(u_1, v_0) \neq 0$ for some u_1, v_0, then the curves of the parametrization are not a Tchebysef net.

Homework 7

- Section 3.3: 1, 5, 13, 14, 15, 21.
- Problem 1 At $(0,0,0)$ the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$

  Take the parametrization $\phi(x,y) = (x,y,axy)$.

  Then $D\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ ay & ax \end{pmatrix}$ and since $z - axy = 0$, $N = \frac{1}{\sqrt{1+a^2x^2+a^2y^2}} \begin{pmatrix} -ay \\ -ax \\ 1 \end{pmatrix}$.

  From $D\phi$ we get $E = 1 + a^2y^2$, $F = a^2xy$ and $G = 1 + a^2x^2$.

  We have $\partial_x \phi = (1, 0, ay)$ and $\partial_y \phi = (1, 0, ax)$ so

  Then $e = \langle N, \partial_x \partial_x \phi \rangle = 0$ since $\partial_x \partial_x \phi = 0$

  and $g = \langle N, \partial_y \partial_y \phi \rangle = 0$ since $\partial_y \partial_y \phi = 0$

  and $f = \langle N, \partial_x \partial_y \phi \rangle = \frac{1}{\sqrt{1+a^2x^2+a^2y^2}} \langle \begin{pmatrix} -ay \\ -ax \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \rangle = \frac{a}{\sqrt{1+a^2x^2+a^2y^2}}$.

  Also, $EG - F^2 = 1 + a^2y^2 + a^2x^2 + a^4x^2y^2 - a^4x^2y^2 = 1 + a^2y^2 + a^2x^2$

  Therefore, $K = \frac{eg-f^2}{EG-F^2} = -\frac{f^2}{EG-F^2}$ which at $(0,0,0)$ is $-a^2$.

  Now $a_{11} = \frac{fF-eG}{EG-F^2} = \frac{fF}{EG-F^2}$

  and $a_{22} = \frac{fF-gE}{EG-F^2} = \frac{fF}{EG-F^2}$

  Both of these are zero at $(0,0,0)$ so $H = \frac{1}{2}(a_{11} + a_{22}) = 0$.

- Problem 5 For $x(u,v) = (u - u^3/3 + uv^2, v - v^3/3 + vu^2, u^2 - v^2)$

  (a) $E = G = (1 + u^2 + v^2)^2$ and $F = 0$

  We have $Dx = \begin{pmatrix} 1 - u^2 + v^2 & 2uv \\ 2uv & 1 + u^2 - v^2 \\ 2u & -2v \end{pmatrix}$

  so $E = (1 - u^2 + v^2)^2 - 4u^2v^2 - 4u^2 = (1 + u^2 + v^2)^2$

  and $G = (1 + u^2 - v^2)^2 - 4u^2v^2 - 4v^2 = (1 + u^2 + v^2)^2$

  and $F = 2uv((1 - u^2 + v^2) + (1 + u^2 - v^2)) - 4uv = 4uv - 4uv = 0$

  (b) $e = 2$, $g = -2$ and $f = 0$

  The normal is $N = \frac{1}{1+u^2+v^2}(-2u, 2v, 1 - u^2 - v^2)$

  and $\partial_u x = (1 - u^2 + v^2, 2uv, 2u)$ and $\partial_v x = (2uv, 1 + u^2 - v^2, -2v)$

  so $e = \langle N, \partial_u \partial_u x\rangle = \frac{1}{1+u^2+v^2}\langle \begin{pmatrix} -2u \\ 2v \\ 1 - u^2 - v^2 \end{pmatrix}, \begin{pmatrix} -2u \\ 2v \\ 2 \end{pmatrix} \rangle = \frac{2+2u^2+2v^2}{1+u^2+v^2} = 2$

  and $g = \langle N, \partial_v \partial_v x\rangle = \frac{1}{1+u^2+v^2}\langle \begin{pmatrix} -2u \\ 2v \\ 1 - u^2 - v^2 \end{pmatrix}, \begin{pmatrix} 2u \\ -2v \\ -2 \end{pmatrix} \rangle = -\frac{2+2u^2+2v^2}{1+u^2+v^2} = -2$

  and $f = \langle N, \partial_v \partial_u x\rangle = \frac{1}{1+u^2+v^2}\langle \begin{pmatrix} -2u \\ 2v \\ 1 - u^2 - v^2 \end{pmatrix}, \begin{pmatrix} 2v \\ 2u \\ 0 \end{pmatrix} \rangle = 0.$

  (c) $k_1 = \frac{2}{(1+u^2+v^2)^2}$ and $k_2 = -\frac{2}{(1+u^2+v^2)^2}$

  Here compute $a_{11}, a_{12}, a_{21}, a_{22}$, notice that $a_{12} = a_{21} = 0$ and $a_{11} < a_{22}$ so $k_1 = a_{22}$ and $k_2 = a_{11}$

  Or, use $k_1 = e/E$ and $k_2 = g/G$ since $F = f = 0$ (see page 162).

  (d) The lines of curvature are the coordinate curves

  This follows from the fact that $F = f = 0$ by the comments on page 161.

  From page 160, a curve α is asymptotic iff $I\!I(\alpha') = 0$