odd multiply perfect numbers that depended on the lower bound we initially imposed on $A$.

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# A Generalized Parallelogram Law

**Alan Nash**

**1. INTRODUCTION.** In a recent note, Quadrat, Lasserre, and Hiriat-Urruty [**6**] prove a theorem for areas of the faces of an orthogonal $n$-simplex that is analogous to the classical theorem of Pythagoras concerning the lengths of the edges of a right triangle. Other, even more general, versions of this result have been published (see [**4**, p. 184], [**2**], and [**5**]).

A generalization of the Pythagorean theorem is the parallelogram law, which in vector language is the formula

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2,$$

and in geometric language is the statement that the sum of the squares of the lengths of the diagonals of a parallelogram is the sum of the squares of the lengths of its sides (the Pythagorean theorem is the special case of rectangles).

The goal of this note is to prove a generalization of the parallelogram law to $n$-dimensional parallelepipeds in the spirit of the generalized Pythagorean theorems referred to in the opening paragraph: "$2^{n-1}$ times the sum of the squares of the

$(n-1)$-dimensional volumes of the faces of an $n$-dimensional parallelepiped is equal to $[(n-1)!\,]^2$ times the sum of the squares of the $(n-1)$-dimensional volumes of its diagonal faces."

**2. NOTATION.** A precise statement of the theorem requires notation for an $n$-parallelepiped and its diagonals. For general information on polytopes, of which $n$-parallelepipeds are particular examples, see [**1**] and [**3**].

Let $\{a_1, \ldots, a_n\}$ (for $n \geq 2$) be a linearly independent subset of $\mathbb{R}^d$, let $A = [\,a_1 \quad \cdots \quad a_n\,]$ be the $d \times n$ matrix of rank $n$ with columns $a_i$, and let $p$ be a point of $R^d$. The *$n$-parallelepiped* with base vertex $p$ determined by $A$ is the set

$$P(A, p) = \left\{ p + \sum_{i=1}^{n} t_i a_i : 0 \leq t_i \leq 1 \text{ for } 1 \leq i \leq n \right\},$$

and the *$n$-simplex* with base vertex $p$ determined by $A$ is the set

$$\begin{aligned}\Delta(A, p) &= \left\{ p + \sum_{i=1}^{n} t_i a_i : t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1 \right\} \\ &= \text{the convex hull of } \{p, p + a_1, \ldots, p + a_n\}.\end{aligned}$$

First, we list a few simple observations about $P(A, p)$ and $\Delta(A, p)$:

1. The vertices of $P(A, p)$ are the points $p_S = p + \sum_{i \in S} a_i$ obtained as $S$ ranges over all subsets of $\{1, \ldots, n\}$. We set $p_\emptyset = p$. There are $2^n$ vertices of $P(A, p)$.
2. The vertices of $\Delta(A, p)$ are $p, p + a_1, \ldots, p + a_n$.
3. If $V_n$ denotes $n$-dimensional volume on an $n$-dimensional affine subspace of $\mathbb{R}^d$, then (see [**5**])

   (a) $V_n(P(A, p)) = |\det A| = \sqrt{A^T A}$

   and

   (b) $V_n(\Delta(A, p)) = \frac{1}{n!} V_n(P(A, p))$.
4. If $A(k)$ denotes the $d \times (n-1)$ matrix obtained by deleting the $k$th column of $A$, then the $n$ faces of $P(A, p)$ adjacent to the vertex $p$ are the $(n-1)$-parallelepipeds $P(A(k), p)$ for $1 \leq k \leq n$.
5. If $S$ and $T$ are subsets of $\{1, \ldots, n\}$, then we say that the vertices $p_S$ and $p_T$ are *adjacent* if $p_T - p_S = \pm a_j$. That is, $p_T$ is reached from $p_S$ by adding or subtracting a vector $a_j$. Notice that $p_T$ and $p_S$ are adjacent if and only if $T$ and $S$ differ by exactly one element, i.e., if and only if $|T \bowtie S| = 1$, where $\bowtie$ denotes symmetric set difference. Hence if $p_S$ is a given vertex of $P(A, p)$, there are exactly $n$ adjacent vertices $p_{S_1}, \ldots, p_{S_n}$, where $S_i$ is $S \bowtie \{i\}$. Notice that $p_{S_i} = p_S + \varepsilon_S(i) a_i$ with

   $$\varepsilon_S(i) = \begin{cases} -1 & \text{if } i \in S, \\ 1 & \text{if } i \notin S. \end{cases}$$

   For example, if $S = \emptyset$ (in which case $p_S = p$), then $p_{S_i} = p + a_i$ for $i = 1, 2, \ldots, n$, while if $p_S = p + a_1$, then $p_{S_1} = p$ and $p_{S_i} = p + a_1 + a_i$ for $i = 2, 3, \ldots, n$.

Given a vertex $p_S$ we define the *diagonal face $D_S$* opposite $p_S$ to be the $(n-1)$-simplex

$$\text{Convex Hull } \{p_{S_1}, \ldots, p_{S_n}\} = \Delta(A_S, p_{S_n}),$$

where

$$\begin{aligned} A_S &= [\, p_{S_1} - p_{S_n} \quad \cdots \quad p_{S_{n-1}} - p_{S_n} \,] \\ &= [\, \varepsilon_S(1)a_1 - \varepsilon_S(n)a_n \quad \cdots \quad \varepsilon_S(n-1)a_{n-1} - \varepsilon_S(n)a_n \,]. \end{aligned}$$

For example, if $p = 0$, then the diagonal opposite the vertex $0 = p_\emptyset$ is the standard $(n-1)$-simplex generated by the columns of $A$. In particular, if $n = 2$, then $D_\emptyset$ is the line segment joining $a_1$ and $a_2$, which has direction vector $a_1 - a_2$, while the diagonal $D_{\{1\}}$ opposite $a_1 = p_{\{1\}}$ is the line segment joining 0 and $a_1 + a_2$, so our definition agrees in this case with the ordinary definition of a diagonal of a parallelogram.

## 3. THE RESULT.

With the notational preliminaries out of the way, we can state the generalized parallelogram law for areas.

**Theorem 1.** *If $A$ is a $d \times n$ matrix of rank $n$ and $p$ is a point of $\mathbb{R}^d$, then*

$$2^n \sum_{k=1}^{n} (V_{n-1}(P(A(k), p)))^2 = \sum_{S \subseteq \{1,\dots,n\}} (V_{n-1}(P(A_S, p_n)))^2 . \qquad (1)$$

**Remarks.** By item 3(b) we have

$$\sum_{S \subseteq \{1,\dots,n\}} (V_{n-1}(P(A_S, p_n)))^2 = [(n-1)!\,]^2 \sum_{S \subseteq \{1,\dots,n\}} (V_{n-1}(D_S))^2 ,$$

which yields the statement in quotation marks given in the introduction.

Also note:

- If $n = 2$, then $V_{n-1}$ is length and equation (1) says

  $$4(\|a_2\|^2 + \|a_1\|^2) = 2(\|a_1 - a_2\|^2 + \|a_1 + a_2\|^2),$$

  which is the ordinary parallelogram law.
- If the columns of $A$ are orthogonal, then all of the diagonal faces of $P(A, p)$ have the same $(n-1)$-volume. Since there are $2^n$ of them, equation (1) becomes (after dividing by $2^n$ and by $(n-1)!^2$ to get volumes of $n$-simplexes, rather than $n$-parallelepipeds)

  $$\sum_{k=1}^{n} (V_{n-1}(\Delta(A(k), p)))^2 = (V_{n-1}(\Delta(A_\emptyset, p_{S_n})))^2 = (V_{n-1}(D_\emptyset))^2 . \qquad (2)$$

  We see that equation (2) is the generalized Pythagorean theorem of [**6**].

  The volume equation follows from the fact that

  $$\|p_{S_i} - p_{S_n}\|^2 = \|p_{S_i} - p_S\|^2 + \|p_{S_n} - p_S\|^2 = \|\varepsilon_S(i)a_i\|^2 + \|\varepsilon_S(n)a_n\|^2 = a_i^2 + a_n^2,$$

  where the first equation is by the Pythagorean theorem and the second one by observation 5. Therefore, in this case $|\det A_S|$ is independent of $S$, as is $V_{n-1}(D_S) = V_{n-1}(\Delta(A_S, p_n))$.

*Proof.* In light of item 3(a), the theorem is an immediate consequence of the following determinant identity, which may be of independent interest.

$$2^n \sum_{k=1}^{n} \det\left(A(k)^T A(k)\right) = \sum_{S\subseteq\{1,\ldots,n\}} \det\left(A_S^T A_S\right). \tag{3}$$

Since the element in row $i$ and column $j$ of $A(k)^T A(k)$ is $a_i \cdot a_j$, the left-hand side of equation (3) is

$$2^n \sum_{k=1}^{n} \sum_{\sigma} \prod_{i=1, i\neq k}^{n} (a_i \cdot a_{\sigma_i}) \operatorname{sgn}(\sigma),$$

where $\sigma$ ranges over all permutations of the set $\{1, 2, \ldots, n\} - \{k\}$ (and where, as usual, $\operatorname{sgn}\sigma$ is 1 if $\sigma$ is a product of an even number of transposition, $-1$ otherwise). This gives $n \cdot (n-1)! = n!$ terms. In each of them, $a_k$ does not appear at all and each of the remaining $a_i$s $(i \neq k)$ appears exactly once on the left of a dot product and exactly once on the right of a dot product.

The right side of equation (3) is

$$\sum_{S\subseteq\{1,2,\ldots,n\}} \sum_{\sigma} \prod_{i=1}^{n-1} \operatorname{sgn}(\sigma)(\varepsilon_S(i)a_i - \varepsilon_S(n)a_n) \cdot (\varepsilon_S(\sigma_i)a_{\sigma_i} - \varepsilon_S(n)a_n),$$

where $\sigma$ ranges over all permutations of the set $\{1, 2, \ldots, n-1\}$.

Since $(\varepsilon_S(i)a_i - \varepsilon_S(n)a_n) \cdot (\varepsilon_S(\sigma_i)a_{\sigma_i} - \varepsilon_S(n)a_n)$ is equal to

$$\varepsilon_S(i)\varepsilon_S(\sigma_i)a_i \cdot a_{\sigma_i} - \varepsilon_S(i)\varepsilon_S(n)a_i \cdot a_n - \varepsilon_S(n)\varepsilon_S(\sigma_i)a_n \cdot a_{\sigma_i} + a_n \cdot a_n$$

for each permutation $\sigma$ and subset $S$, the product

$$\prod_{i=1}^{n-1} \operatorname{sgn}(\sigma)(\varepsilon_S(i)a_i - \varepsilon_S(n)a_n) \cdot (\varepsilon_S(\sigma_i)a_{\sigma_i} - \varepsilon_S(n)a_n)$$

gives $4^{n-1}$ terms. Each of these terms $t_{S,\sigma}$ is a product of $\operatorname{sgn}(\sigma)$ and $n-1$ factors, each of them having one of the four forms

$$\varepsilon_S(i)\varepsilon_S(\sigma_i)a_i \cdot a_{\sigma_i}, \quad -\varepsilon_S(i)\varepsilon_S(n)a_i \cdot a_n, \quad -\varepsilon_S(n)\varepsilon_S(\sigma_i)a_n \cdot a_{\sigma_i}, \quad a_n \cdot a_n,$$

and one for each $i$ satisfying $1 \le i \le n-1$. Since we have $2^n$ subsets and $(n-1)!$ permutations, we get a total of $2^n(n-1)!\,4^{n-1}$ terms. Fortunately, most of them cancel each other.

In each term, $a_i$ $(1 \le i \le n-1)$ clearly can appear at most once on each side of a dot product: on the left in a factor of the form $\varepsilon_S(i)\varepsilon_S(\sigma_i)a_i \cdot a_{\sigma_i}$ or $-\varepsilon_S(i)\varepsilon_S(n)a_i \cdot a_n$ or on the right in a factor of the form $\varepsilon_S(j)\varepsilon_S(\sigma_j)a_j \cdot a_{\sigma_j}$ or $-\varepsilon_S(n)\varepsilon_S(\sigma_j)a_n \cdot a_{\sigma_j}$ for $j$ such that $\sigma_j = i$.

Any term $t_{S,\sigma}$ with some $a_i$ $(1 \le i \le n-1)$ appearing only on the left or only on the right of a dot product will be cancelled by the corresponding term $t_{S_i,\sigma}$, obtained from $t_S$ by replacing every occurrence of $S$ by $S_i$, because $\varepsilon_S(i) = -\varepsilon_{S_i}(i)$. We are left only with terms in which every $a_i$ that appears does so on both the left and the right of a dot product.

Any term $t_{S,\sigma}$ containing a factor of the form $-\varepsilon_S(i)\varepsilon_S(n)a_i \cdot a_n$ and a factor of the form $-\varepsilon_S(j)\varepsilon_S(n)a_j \cdot a_n$ or a factor of the form $-\varepsilon_S(n)\varepsilon_S(\sigma_i)a_n \cdot a_{\sigma_i}$ and a factor of the form $-\varepsilon_S(n)\varepsilon_S(\sigma_j)a_n \cdot a_{\sigma_j}$ for $j \neq i$ and $1 \leq i, j \leq n$ will be cancelled by the corresponding term $t_{S,\tau}$, where $\tau$ is equal to $\sigma$ except that $\tau_i = \sigma_j$ and $\tau_j = \sigma_i$ (notice that $\operatorname{sgn}\sigma = -\operatorname{sgn}\tau$).

Therefore, each remaining term must have all $a_i$s $(1 \leq i \leq n)$ except for one appearing once on the left of a dot product and once on the right. Each of these remaining terms is a product of $\operatorname{sgn}(\sigma)$ and $n-1$ factors, each factor having one of the forms

$$a_i \cdot a_{\sigma_i}, \quad a_i \cdot a_n, \quad a_n \cdot a_{\sigma_i}, \quad a_n \cdot a_n,$$

since all $\varepsilon_S(k)$ occur in pairs. Notice that these terms do not depend on $S$.

Therefore, for any subset $S$ of $\{1, 2, \dots n\}$ we have a bijection between equal terms in

$$\sum_{k=1}^{n} \sum_{\sigma} \prod_{i=1, i\neq k}^{n} (a_i \cdot a_{\sigma_i}) \operatorname{sgn}(\sigma), \tag{4}$$

where $\sigma$ ranges over all permutations of the set $\{1, 2, \dots, n\} - \{k\}$, and

$$\sum_{S\subseteq\{1,2,\dots,n\}} \sum_{\sigma} \prod_{i=1}^{n-1} \operatorname{sgn}(\sigma)(\varepsilon_S(i)a_i - \varepsilon_S(n)a_n) \cdot (\varepsilon_S(\sigma_i)a_{\sigma_i} - \varepsilon_S(n)a_n), \tag{5}$$

where $\sigma$ ranges over all permutations of the set $\{1, 2, \dots, n-1\}$, once the terms in (5) that we have shown to cancel in pairs are removed. Any term $t_{S,\sigma}$ in (5) not containing $a_n$ corresponds to a term in (4) for $k = n$ (and conversely) under the same permutation $\sigma$. Any term $t_{S,\sigma}$ in (5) not containing $a_k$ $(1 \leq k \leq n-1)$ corresponds to a term in (4) for $k \neq n$ (and conversely) under the permutation $\tau$ of the set $\{1, 2, \dots, n\} - \{k\}$ given by

$$\tau(i) = \begin{cases} n & \text{if } \sigma(i) = k, \\ \sigma(k) & \text{if } i = n, \\ \sigma(i) & \text{otherwise} \end{cases}$$

(notice that $\operatorname{sgn}\sigma = \operatorname{sgn}\tau$).

Therefore (4) and (5) are equal. Multiplying (4) by $2^n$ is the same as summing (5) over all $2^n$ subsets $S$ of $\{1, 2, \dots n\}$, so the left and right sides of equation (3) are equal. ■

Note that the matrices in equation (3) are formed from a given matrix $A$ by ordinary matrix operations, and hence the identity is valid for matrices with entries in any commutative ring $R$. Of course, the interpretation as volumes is only valid if $R = \mathbb{R}$.

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# Rearrangement of a Conditionally Convergent Series

## Uri Elias

One of the surprising results in an elementary calculus course is that a rearrangement of a conditionaly convergent series may change its sum, even its very convergence. Observing typical textbook examples of this phenomenon, it turns out that during the rearrangement some of the terms are moved arbitrarily large distances from their original locations. Is this necessary? The answer is positive. Indeed, we can assert:

*Let $\sum a_n$ be a convergent series. Suppose that, in a rearrangement of the series, there is a fixed positive integer $p$ such that each term of the series that is shifted forward is shifted at most $p$ places. Then the rearranged series converges to the same sum as the original one.*

Note that there is a clear difference between forward shifts and backward shifts.

The proof is straightforward. Let $S_n = a_1 + \cdots + a_n$ and $T_n = a_{\pi(1)} + \cdots + a_{\pi(n)}$ be the partial sums of the original and rearranged series, respectively, where $\pi$ denotes the corresponding permutation of the positive integers. Then $T_n$ consists of the terms of $S_n$, with the possible exception of $2p$ terms: some of the last $p$ terms of $S_n$ could be moved forward by $\pi$ and excluded from $T_n$, in which event they would be replaced by at most $p$ terms $a_{n_1}, \ldots, a_{n_p}$, with $n < n_1 < \cdots < n_p$, that are moved backward, possibly from arbitrarily large distances. Thus,

$$|T_n - S_n| \le |a_{n-p+1}| + \cdots + |a_n| + |a_{n_1}| + \cdots + |a_{n_p}|.$$

Due to the convergence of $\sum a_n$, for each $\varepsilon > 0$ it is the case that $|a_k| < \varepsilon/2p$ once $k > N(\varepsilon)$. Consequently, $|T_n - S_n| < \varepsilon$ whenever $n > N(\varepsilon) + p$. ∎

This proposition intends to give the students the feeling that conditionaly convergent series, as well as math professors, may tolerate perturbations, as long as they remain bounded. However, both series and professors may undergo surprising metamorphoses when the perturbations go on and on.

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